

Navier–Stokes limit of elastic bilinear kinetic equations

Joint work with Bertrand Lods (Universita degli studi di Torino)

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(BCAM)

Introduction

Collisional kinetic equation with no force :

$$(\partial_t + v \cdot \nabla_x)F(t, x, v) = \mathcal{C}[F(t, x, \cdot)](v)$$

Macroscopic elastic conservation (mass, momentum, energy):

$$\varphi(v) = 1, v_1, \dots, v_d, |v|^2, \quad \int_{\mathbb{R}^d} \mathcal{C}[F](v)\varphi(v)dv = 0.$$

Microscopic dissipation (H-theorem) : For some convex $\Phi : \mathbb{R} \rightarrow \mathbb{R}$

$$\forall F, \quad \int_{\mathbb{R}^d} \mathcal{C}[F](v)\Phi(F(v))dv \leq 0.$$

and, denoting $\mathbf{m} = \int_{\mathbb{R}^d} F(v) (1, v, |v|^2) dv$ the conserved macroscopic quantities

$$\forall F, \quad \int_{\mathbb{R}^d} \mathcal{C}[F](v)\Phi(F(v))dv = 0 \Leftrightarrow \mathcal{C}[F] = 0 \Leftrightarrow F(v) = \mathcal{M}_{\mathbf{m}}(v).$$

Examples : (quantum) Boltzmann and Landau, BGK, non-linear Fokker-Planck

Introduction

Small fluctuations in diffusive regime : $F(t, x, v) = \mathcal{M}_{1,0,1}(v) + \varepsilon f^\varepsilon (\varepsilon^2 t, \varepsilon x, v)$

$$\partial_t f^\varepsilon = \varepsilon^{-2}(\mathcal{L} - \varepsilon v \cdot \nabla_x) f^\varepsilon + \varepsilon^{-1} \mathcal{Q}(f^\varepsilon, f^\varepsilon) + o(\varepsilon^{-1})$$

where $\mathcal{C}[\mathcal{M}_{1,0,1} + \varepsilon g] = \varepsilon \mathcal{L}g + \varepsilon^2 \mathcal{Q}(g, g) + o(\varepsilon^2)$

Theorem (Bardos, Golse, Levermore, 1991 (for $\Phi(u) = -\log u$))

If $f^\varepsilon \rightarrow f^0$ and every term of the equation converges formally, then

$$f^0(t, x, v) = (u^0(t, x) \cdot v + \theta^0(t, x)(|v|^2 - cst.)) \mathcal{M}_{1,0,1}(v)$$

$$\begin{cases} \partial_t u^0 + u^0 \cdot \nabla u^0 = \kappa_{inc} \Delta u^0 + \nabla p, & \nabla \cdot u^0 = 0, & \kappa_{inc} = \kappa_{inc}(\mathcal{L}) > 0, \\ \partial_t \theta^0 + u^0 \cdot \nabla \theta^0 = \kappa_{Fou} \Delta \theta^0, & \kappa_{Fou} = \kappa_{Fou}(\mathcal{L}) > 0 \end{cases}$$

Weak convergence: For Boltzmann (Bardos, Golse, Levermore, Saint-Raymond, Arsénio)

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Quantitative convergence: For Boltzmann or Landau (Bardos, Ukai, Gallagher, Tristani, Carrapatoso, Rachid) using spectral studies (Ellis, Pinsky, Yang, Yu)

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Limitations on quantitative approach:

- ▶ Only $\Phi(u) = -\log u$ (exclude quantum equations)
- ▶ Equations treated separately (although common structure)
- ▶ Non-constructive spectral studies
- ▶ Restrictive integrability condition

Introduction

Our structural assumptions

Consider $V^1 \subset V := L^2(\mu^{-1}dv)$ where $\langle v \rangle^k \mu \in L^1_+$ and $\mathcal{L} = \mathcal{L}^*$

(H1) Isotropy : $\forall O \in \mathbb{R}^{d \times d}$ orthogonal, denoting $(Of)(v) = f(Ov)$

$$O(\mathcal{L}f) = \mathcal{L}(Of), \quad O\mathcal{Q}(f, f) = \mathcal{Q}(Of, Of),$$

(H2) Macroscopic conservation law:

$$\ker \mathcal{L} = \text{Span}(\mu, v_1\mu, \dots, v_d\mu, |v|^2\mu) \subset V^1$$

$$\langle \mathcal{L}f, \varphi \rangle_V = \langle \mathcal{Q}(f, f), \varphi \rangle_V = 0, \quad \varphi \in \ker(\mathcal{L})$$

(H3) Microscopic dissipation (linearized H-theorem):

$$\langle \mathcal{L}f, f \rangle_V \leq -\|\Pi_{(\ker \mathcal{L})^\perp} f\|_{V^1}^2 \quad \text{and} \quad [\mathcal{L}, \langle v \rangle^{1 \text{ and } 2}] = \text{L.O.T.}$$

~~$\mathcal{L} = \text{dissipative} + \text{compact}$~~

(H4) Control of the collisions by the energy V and the dissipated entropy V^1 :

$$\langle \mathcal{Q}(f, g), h \rangle_V \lesssim \|h\|_{V^1} (\|f\|_{V^1} \|g\|_V + \|f\|_V \|g\|_{V^1})$$

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(H4) Control of the collisions by the energy V and the dissipated entropy V^1 :

$$\|\mathcal{Q}(f, g)\|_{V^{-1}} \lesssim \|f\|_{V^1} \|g\|_V + \|f\|_V \|g\|_{V^1}, \quad V^{-1} := (V^1)' \text{ w.r.t. } V$$

Strategy

Our problem: profile of small fluctuations f^ε around an equilibrium

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f^\varepsilon = \frac{1}{\varepsilon^2} \mathcal{L} f^\varepsilon + \frac{1}{\varepsilon} \mathcal{Q}(f^\varepsilon, f^\varepsilon) + \mathcal{O}(\varepsilon^{-1}), \quad f^\varepsilon(0, x, v) = f_{\text{in}}(x, v)$$

Goal: show that

$$f^\varepsilon(t, x, v) \xrightarrow{\varepsilon \rightarrow 0} (u(t, x) \cdot v + \theta(t, x) (|v|^2 - \text{cst.})) \mu(v)$$

$$\begin{cases} \partial_t u + u \cdot \nabla_x u = \kappa_{\text{inc}} \Delta_x u, & \nabla_x \cdot u = 0, \\ \partial_t \theta + u \cdot \nabla_x \theta = \kappa_{\text{Fou}} \Delta_x \theta. \end{cases}$$

and describe/quantify the convergence.

Strategy

Our problem: profile of small fluctuations f^ε around an equilibrium

$$f^\varepsilon(t) = U^\varepsilon(t)f_{\text{in}} + \Psi^\varepsilon(f^\varepsilon, f^\varepsilon)(t)$$

where

$$U^\varepsilon(t) := \exp\left(\frac{t}{\varepsilon^2}(\mathcal{L} - \varepsilon v \cdot \nabla_x)\right)$$

$$\Psi^\varepsilon(f, f)(t) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - \tau) \mathcal{Q}(f(\tau), f(\tau)) d\tau$$

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Strategy

Strategy for the existence of solutions

Identify some “hydrodynamic” and “kinetic” regimes:

$$\|f\|_{\mathcal{H}}^2 := \sup_{t \geq 0} \|f(t)\|_{H_x^\ell V_v^1}^2 + \int_0^\infty \|\nabla_x f(t)\|_{H_x^\ell V_v^1}^2 dt, \quad \begin{array}{l} \text{(heat equation)} \\ \text{(Navier-Stokes)} \end{array}$$

$$\|f\|_{\mathcal{K}^\varepsilon}^2 := \sup_{t \geq 0} e^{2\sigma t/\varepsilon^2} \|f(t)\|_{H_x^\ell V_v}^2 + \frac{1}{\varepsilon^2} \int_0^\infty e^{2\sigma t/\varepsilon^2} \|f(t)\|_{H_x^\ell V_v^1}^2 dt, \quad \begin{array}{l} \text{(dissipative} \\ \text{equation)} \end{array}$$

and a decomposition of the semigroup/nonlinearity:

$$U^\varepsilon(t) = \exp\left(\frac{t}{\varepsilon^2} (\mathcal{L} - \varepsilon v \cdot \nabla_x)\right) = U_{\text{hydro}}^\varepsilon \oplus U_{\text{kine}}^\varepsilon$$

$$\Psi^\varepsilon(f, f)(t) = \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t - \tau) \mathcal{Q}(f(\tau), f(\tau)) d\tau = \Psi_{\text{hydro}}^\varepsilon(f, f) + \Psi_{\text{kine}}^\varepsilon(f, f)$$

compatible with the corresponding regimes:

$$\begin{aligned} \|U_{\text{hydro}}^\varepsilon f\|_{\mathcal{H}} &\lesssim \|f\|_{H_x^\ell V_v}, & \|U_{\text{kine}}^\varepsilon f\|_{\mathcal{K}^\varepsilon} &\lesssim \|f\|_{H_x^\ell V_v} \\ \|\Psi_{\text{hydro}}^\varepsilon(f, f)\|_{\mathcal{H}} &\lesssim \|f\|_{\mathcal{H}}^2, & \|\Psi_{\text{kine}}^\varepsilon(f, f)\|_{\mathcal{K}^\varepsilon} &\lesssim \|f\|_{\mathcal{K}^\varepsilon}^2 \end{aligned}$$

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Consider an **arbitrary** system of equations

$$\left. \begin{array}{l} f_{\text{hydro}}^\varepsilon(t) = U_{\text{hydro}}^\varepsilon(t) f_{\text{in}} + \Psi_{\text{hydro}}^\varepsilon(f_{\text{hydro}}^\varepsilon, f_{\text{hydro}}^\varepsilon) + \dots \\ f_{\text{kine}}^\varepsilon(t) = U_{\text{kine}}^\varepsilon(t) f_{\text{in}} + \Psi_{\text{kine}}^\varepsilon(f_{\text{kine}}^\varepsilon, f_{\text{kine}}^\varepsilon) + \dots \\ \dots \end{array} \right\} \xrightarrow{\text{Picard}} \begin{array}{l} f^\varepsilon := f_{\text{hydro}}^\varepsilon + f_{\text{kine}}^\varepsilon + \dots \\ \in \mathcal{H} + \mathcal{K}^\varepsilon + \dots \\ \text{is a solution} \end{array}$$

Strategy

Strategy for the convergence

Kinetic convergence : OK because $\|f_{\text{kin}}^\varepsilon\|_{\mathcal{K}^\varepsilon} \lesssim 1 \Rightarrow f_{\text{kin}}^\varepsilon = \mathcal{O}\left(e^{-\sigma t/\varepsilon^2}\right)$

Hydrodynamic convergence : To prove the convergence of

$$f_{\text{hydro}}^\varepsilon(t) = U_{\text{hydro}}^\varepsilon(t) f_{\text{in}} + \Psi_{\text{hydro}}^\varepsilon(f_{\text{hydro}}^\varepsilon, f_{\text{hydro}}^\varepsilon) + \dots$$

(1) Show the convergence of the “hydrodynamic” operators:

$$U_{\text{hydro}}^\varepsilon = U_{\text{NS}}^0 + \mathcal{O}(\varepsilon) + \text{dispersive} \quad \text{and} \quad \Psi_{\text{hydro}}^\varepsilon(f, f) = \Psi_{\text{NS}}^0(f, f) + \mathcal{O}(\varepsilon)$$

(2) Check from **explicit** expressions of U_{NS}^0 and Ψ_{NS}^0 that

$$f(t) = U_{\text{NS}}^0(t) f(0) + \Psi_{\text{NS}}^0(f, f)(t)$$
$$\Leftrightarrow \begin{cases} f(t, x, v) = (u(t, x) \cdot v + \theta(t, x) (|v|^2 - \text{cst.})) \mu(v), \\ (u, \theta) \text{ is a mild solution of INSF} \end{cases}$$

Outline of the proof of the (non)linear bounds

Proposition (G, Lods)

The semigroup and Duhamelized nonlinearity split

$$U^\varepsilon(t) = \exp\left(\frac{t}{\varepsilon^2}(\mathcal{L} - \varepsilon v \cdot \nabla_x)\right) = U_{\text{hydro}}^\varepsilon(t) \oplus U_{\text{kine}}^\varepsilon(t)$$

$$\frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-\tau) \mathcal{Q}(f(\tau), f(\tau)) d\tau = \Psi_{\text{hydro}}^\varepsilon(f, f)(t) + \Psi_{\text{kine}}^\varepsilon(f, f)(t)$$

and satisfy the continuity estimates

$$\|U_{\text{hydro}}^\varepsilon f\|_{\mathcal{H}} \lesssim \|f\|_{H_x^\ell V_v}, \quad \|U_{\text{kine}}^\varepsilon f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H_x^\ell V_v}$$

$$\|\Psi_{\text{hydro}}^\varepsilon(f, f)\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2, \quad \|\Psi_{\text{kine}}^\varepsilon(f, f)\|_{\mathcal{K}^\varepsilon} \lesssim \varepsilon \|f\|_{\mathcal{K}^\varepsilon}^2$$

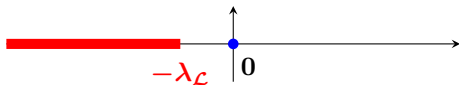
Difficulties:

1. Find the decomposition in stable subspaces $H_x^\ell V_v = \text{hydro}^\varepsilon \oplus \text{kine}^\varepsilon$
2. Compensate the stiff factor $\frac{1}{\varepsilon}$
3. Compensate the deregularizing effect $\mathcal{Q} : V^1 \rightarrow V^{-1}$

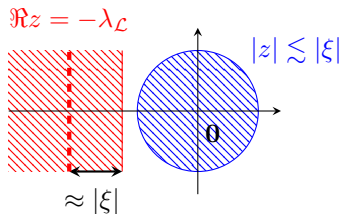
Outline of the proof of the (non)linear bounds

Finding stable subspaces of $\mathcal{L} - v \cdot \nabla_x$: localization of $\Sigma(\mathcal{L} - iv \cdot \xi)$

Localization for $\xi = 0$:

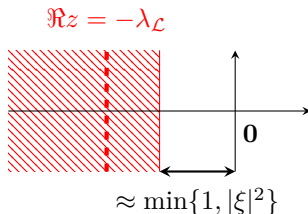


Localization for $|\xi| \ll 1$:



Factorization method inspired from [T]
using $[\mathcal{L}, \langle v \rangle] = \text{L.O.T.}$ from **(H3)**

Localization for $|\xi| \gtrsim 1$:



Mode by mode hypocoercivity from [D]

Disjoint parts of $\Sigma(\mathcal{L} - iv \cdot \xi) \Rightarrow$ stable **kinetic**(ξ) and **hydro**(ξ) subspaces

Outline of the proof of the (non)linear bounds

Localization of hydrodynamic modes for $|\xi| \ll 1$ and $\Re z \geq -\lambda_{\mathcal{L}}$

Naive approach for localization:

$$(\mathcal{L} - iv \cdot \xi - z)^{-1} = (\mathcal{L} - z)^{-1} + (\mathcal{L} - iv \cdot \xi - z)^{-1} \underbrace{(iv \cdot \xi)}_{\text{unbounded!}} (\mathcal{L} - z)^{-1} \quad (\star)$$

Solution by enlargement method: $\mathcal{L} = \mathcal{B} + \mathcal{A}$ with $\mathcal{A} = M\mathbf{1}_{|v| \leq M}$

$$\begin{aligned} [\mathcal{L}, \langle v \rangle] = \text{L.O.T. from (H3)} &\Rightarrow \mathcal{B}_{\xi} := \mathcal{B} - iv \cdot \xi \text{ dissipative on } L^2(\langle v \rangle \mu^{-1} dv) \\ &\Rightarrow \|(\mathcal{L} - z)^{-1}\| \lesssim 1 + 1/|z| \text{ on } L^2(\langle v \rangle \mu^{-1} dv) \end{aligned}$$

Insert (\star) in the right hand side of the expansion

$$(\mathcal{L} - iv \cdot \xi - z)^{-1} = (\mathcal{B}_{\xi} - z)^{-1} + (\mathcal{L} - iv \cdot \xi - z)^{-1} \mathcal{A} (\mathcal{B}_{\xi} - z)^{-1}$$

to use that \mathcal{A} compensates the unboundedness of v :

$$\begin{aligned} (\mathcal{L} - iv \cdot \xi - z)^{-1} &= \underbrace{(\mathcal{B}_{\xi} - z)^{-1} + (\mathcal{L} - z)^{-1} \mathcal{A} (\mathcal{B}_{\xi} - z)^{-1}}_{\text{bounded}} \\ &\quad + (\mathcal{L} - iv \cdot \xi - z)^{-1} \underbrace{(iv \cdot \xi) (\mathcal{L} - z)^{-1} \mathcal{A} (\mathcal{B}_{\xi} - z)^{-1}}_{\text{of order } |\xi|(1+1/|z|)} \end{aligned}$$

Thus bounded for $|\xi| \ll 1$ and $|\xi| \ll |z| \Rightarrow$ spectrum localized in $B(0, C|\xi|)$.

Outline of the proof of the (non)linear bounds

Localization of hydrodynamic modes for $|\xi| \ll 1$ and $\Re z \geq -\lambda_{\mathcal{L}}$

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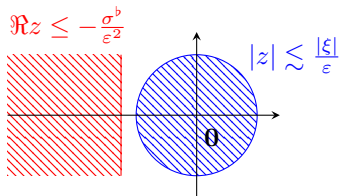
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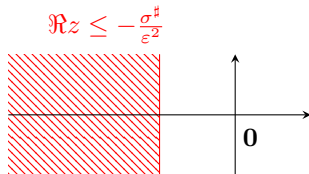
The kinetic regime

$$\text{Scaling } (t, x) \rightarrow \left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) \Rightarrow \mathcal{L} - i(v \cdot \xi) \rightarrow \frac{1}{\varepsilon^2} (\mathcal{L} - i\varepsilon(v \cdot \xi))$$

Localization for $|\xi| \ll \varepsilon^{-1}$:



Localization for $|\xi| \gtrsim \varepsilon^{-1}$:



$$f \in V \xrightarrow{\text{localization}} U_{\text{kine}}^\varepsilon f \in L_t^\infty V_v \left(e^{t \frac{\sigma}{\varepsilon^2}} dt \right) \xrightarrow[\text{using (H3)}]{\text{energy method}} \underbrace{\frac{1}{\varepsilon} U_{\text{kine}}^\varepsilon f \in L_t^2 \mathbf{V}_v^1 \left(e^{t \frac{\sigma}{\varepsilon^2}} dt \right)}_{(*)}$$

$$\Rightarrow \boxed{\|U_{\text{kine}}^\varepsilon f\|_{\mathcal{K}^\varepsilon} \lesssim \|f\|_{H_x^l V_v}} \xrightarrow[\text{= double regularization, double smallness}]{\text{convolution using } (*)} \boxed{\underbrace{\left\| \frac{1}{\varepsilon} U_{\text{kine}}^\varepsilon *_t \mathcal{Q}(f, f) \right\|_{\mathcal{K}^\varepsilon}}_{= \|\Psi_{\text{kine}}^\varepsilon(f, f)\|_{\mathcal{K}^\varepsilon}} \lesssim \varepsilon \|f\|_{\mathcal{K}^\varepsilon}^2}$$

Outline of the proof of the (non)linear bounds

Construction of the hydrodynamic modes

Expansion of the spectral hydrodynamic projector/Kato's isomorphism :

- Resolvent bounds + enlargement method using $[\mathcal{L}, \langle v \rangle^2 \times] = \text{L.O.T.}$ from **(H3)**
- \Rightarrow expansion of hydrodynamic projector $\mathcal{P}(\xi) : V^{-1} \rightarrow \text{hydro space}(\xi) \subset V^1$
 - \Rightarrow expansion of Kato's isomorphism $\Phi(\xi) : \ker(\mathcal{L}) \rightarrow \text{hydro space}(\xi)$

Diagonalization/expansion of the hydrodynamic part of $\mathcal{L} - iv \cdot \xi$ for $|\xi| \ll 1$:

- Define the reduced matrix $L(\xi) \in \mathcal{B}(\ker(\mathcal{L})) \stackrel{\Phi(\xi)}{\sim} (\mathcal{L} - i(v \cdot \xi))|_{\text{hydro. space}(\xi)}$
- Isotropy $\Rightarrow L(\xi)|_{|\xi|=0}$ diagonalizable \Rightarrow diagonalization/expansion of $L(\xi)$

\Rightarrow expansion of spectral projections using expansion of $\Phi(\xi)$:

$$\mathcal{P}(\xi) = \mathcal{P}_{\text{Fou}}(\xi) \oplus \mathcal{P}_{\text{inc}}(\xi) \oplus \mathcal{P}_{+\text{osc}}(\xi) \oplus \mathcal{P}_{-\text{osc}}(\xi),$$
$$\mathcal{P}_\star(\xi) = \mathcal{P}_\star^{(0)}(\xi/|\xi|) + \xi \cdot \mathcal{P}_\star^{(1)}(\xi/|\xi|) + \mathcal{O}(|\xi|^2) : V_v^{-1} \rightarrow V_v^1$$

\Rightarrow expansion of eigenvalues:

$$\lambda_{\text{Fou}}(\xi) = -\kappa_{\text{Fou}}|\xi|^2 + \mathcal{O}(|\xi|^3), \quad \lambda_{\text{inc}}(\xi) = -\kappa_{\text{inc}}|\xi|^2 + \mathcal{O}(|\xi|^3),$$
$$\lambda_{\pm\text{osc}}(\xi) = \pm ic|\xi| - \kappa_{\text{osc}}|\xi|^2 + \mathcal{O}(|\xi|^3)$$

Outline of the proof of the (non)linear bounds

Hydrodynamic regime ($\star = \text{inc, Fou, } \pm \text{osc}$)

Linear estimate:

$$\lambda_{\star}(\xi) \approx \alpha_{\star}|\xi| - \kappa_{\star}|\xi|^2 \Rightarrow \Re\left(\frac{\lambda_{\star}(\varepsilon\xi)}{\varepsilon^2}\right) \leq -\frac{\kappa_{\star}}{2}|\xi|^2 \quad \text{and} \quad \mathcal{P}_{\star}(\xi) : V_v^{-1} \rightarrow V_v^1$$
$$\Rightarrow \underbrace{\left\| e^{t\frac{\lambda_{\star}(\varepsilon\nabla_x)}{\varepsilon^2}} \mathcal{P}_{\star}(\varepsilon\nabla_x) f \right\|_{H_x^{\ell} V_v^1}}_{\|U_{\star}^{\varepsilon}(t)f\|_{H_x^{\ell} V_v^{-1}}} \lesssim \left\| e^{t\frac{\kappa_{\star}}{2}\Delta_x} f \right\|_{H_x^{\ell} V_v^{-1}} \Rightarrow \boxed{\|U_{\text{hydro}}^{\varepsilon} f\|_{\mathcal{H}} \lesssim \|f\|_{H_x^{\ell} V_v^{-1}}}$$

Reminder: $\mathcal{H} = L_t^{\infty} H_x^{\ell} \cap L_t^2 \dot{H}_x^{\ell+1}$ (parabolic regularity)

Nonlinear estimate:

$$\mathcal{P}^{(0)} \Pi_{\ker(\mathcal{L})} = \mathcal{P}^{(0)} \quad \text{and} \quad \mathcal{Q} \perp \ker(\mathcal{L}) \text{ from (H2)} \Rightarrow \mathcal{P}^{(0)} \mathcal{Q} = 0$$
$$\Rightarrow \frac{1}{\varepsilon} \mathcal{P}_{\star}(\varepsilon\xi) \mathcal{Q} \approx \mathbf{0} + \xi \cdot \mathcal{P}^{(1)}(\xi/|\xi|) \mathcal{Q}(f, g) \in V_v^1$$
$$\Rightarrow \left\| \frac{1}{\varepsilon} U_{\text{hydro}}^{\varepsilon} *_t \mathcal{Q}(f, f) \right\|_{\mathcal{H}} = \boxed{\|\Psi_{\text{hydro}}^{\varepsilon}(f, f)\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2}$$

Rough idea of the proof of hydrodynamic convergence

Refining the previous step, one gets for $\star = \text{Fou, inc}$

$$U_{\star}^{\varepsilon}(t) = U_{\star}^0(t) + \mathcal{O}(\varepsilon) \quad \text{and} \quad \Psi_{\star}^{\varepsilon}(t) = \Psi_{\star}^0(t) + \mathcal{O}(\varepsilon)$$

where each limit writes

$$U^0(t) = e^{t\kappa_{\star}\Delta_x} \mathcal{P}^{(0)}(\nabla_x)$$
$$\Psi_{\star}^0(f, f)(t) = \int_0^t e^{(t-\tau)\kappa_{\star}\Delta_x} \mathcal{P}^{(1)}(\nabla_x) \mathcal{Q}(f(\tau), f(\tau)) \, d\tau$$

which is related to the mild formulation of Navier-Stokes-Fourier:

$$u(t) = e^{t\kappa_{\text{inc}}\Delta_x} u(0) + \int_0^t e^{(t-\tau)\kappa_{\text{inc}}\Delta_x} \mathbb{P} [\nabla_x \cdot (u(\tau) \otimes u(\tau))] \, d\tau$$

$$\theta(t) = e^{t\kappa_{\text{Fou}}\Delta_x} \theta(0) + \int_0^t e^{(t-\tau)\kappa_{\text{Fou}}\Delta_x} [\nabla_x \cdot (\theta(\tau)u(\tau))] \, d\tau$$

And for the oscillating terms:

$$U_{\pm\text{osc}}^{\varepsilon}(t) = \mathcal{O}\left(\frac{\varepsilon}{t}\right) \quad \text{and} \quad \Psi_{\pm\text{osc}}^{\varepsilon}(f, f) = \mathcal{O}(\varepsilon).$$

TLDR: study of the hydrodynamic operators

Perturbation in finite dimension \Rightarrow construction of eigenmodes $(\lambda_\star(\xi), \mathcal{P}_\star(\xi))$

$$\widehat{U_\star^\varepsilon(t)} f(\xi) = e^{\varepsilon^{-2} t \lambda_\star(\varepsilon\xi)} \mathcal{P}_\star(\varepsilon\xi) \widehat{f}(\xi), \quad U_{\text{hydro}}^\varepsilon = U_{\text{inc}}^\varepsilon + U_{\text{Fou}}^\varepsilon + U_{\text{osc}}^\varepsilon$$

- ▶ $\Re(\lambda_\star(\xi)) \lesssim -|\xi|^2 \Rightarrow$ parabolic estimate in (t, x)
- ▶ $\mathcal{P}_\star(\xi) : V^{-1} \rightarrow V^1 \Rightarrow$ regularization in v

$$\|U_\star^\varepsilon f\|_{\mathcal{H}} \lesssim \|f\|_{H_x^\ell V_v^{-1}}$$

- ▶ $\mathcal{P}(\varepsilon\xi) = \mathcal{P}^{(0)}(\xi/|\xi|) + \mathcal{O}(\varepsilon\xi)$
- ▶ $Q \perp \ker(\mathcal{L})$ from **(H2)** + $\mathcal{P}^{(0)} = \mathcal{P}^{(0)} \Pi_{\ker(\mathcal{L})} \Rightarrow \mathcal{P}^{(0)} Q = 0$

$$\Rightarrow \left\| \frac{1}{\varepsilon} U_{\text{hydro}}^\varepsilon *_t Q(f, f) \right\|_{\mathcal{H}} = \|\Psi_{\text{hydro}}^\varepsilon(f, f)\|_{\mathcal{H}} \lesssim \|f\|_{\mathcal{H}}^2$$

Enlargement + $[\mathcal{L}, \langle v \rangle^2] = \text{LOT}$ from **(H3)** \Rightarrow good expansion of $(\lambda_\star(\xi), \mathcal{P}_\star(\xi))$
 \Rightarrow expansion of $U_\star^\varepsilon(t)$ and $\Psi_\star^\varepsilon(f, f)$

$$U_{\text{inc}}^\varepsilon + U_{\text{Fou}}^\varepsilon = U_{\text{NS}} + \mathcal{O}(\varepsilon)$$

$$U_{\text{osc}}^\varepsilon = \mathcal{O}(\varepsilon/t)$$

$$\Psi_{\text{hydro}}^\varepsilon = \Psi_{\text{NS}} + \mathcal{O}(\varepsilon)$$

Conclusion

Statement of the theorem

Previous estimates: $\|f_{\text{hydro}}^\varepsilon(0)\| \ll 1$ and $\varepsilon \ll 1 \xrightarrow{\text{Picard near 0}} \exists f^\varepsilon$.

Better: $\exists f_{\text{NS}}^0$ + norm such that $\|f_{\text{NS}}^0\|_{\mathcal{H}(f_{\text{NS}}^0, \delta)} \leq \delta \ll 1 \xrightarrow{\text{Picard near } f^0} \exists f^\varepsilon$ (as in [GT])

Theorem (G, Lods) — ArXiv:2304.11698

Consider any (**non-small**) $f_{\text{in}} \in H_x^\ell L_v^2(\mu^{-1}dv)$, for any $\varepsilon \ll 1$, $\exists!$ solution to

$$\partial_t f^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x f^\varepsilon = \varepsilon^{-2} \mathcal{L} + \varepsilon^{-1} \mathcal{Q}(f^\varepsilon, f^\varepsilon), \quad f^\varepsilon \in L_t^\infty H_x^\ell L^2(\mu^{-1}dvdx),$$

with the **same lifespan** as the Navier-Stokes limit f^0 , and for smooth initial data

$$f^\varepsilon = f_{\text{NS}}^0 + \mathcal{O}\left(e^{-t\sigma/\varepsilon^2}\right) + \mathcal{O}(\sqrt{\varepsilon}) + \mathcal{O}\left(\left(\frac{\varepsilon}{t}\right)^{\frac{d-1}{2}}\right)$$

and for incompressible initial data

$$f^\varepsilon = f_{\text{NS}}^0 + \mathcal{O}\left(e^{-t\sigma/\varepsilon^2}\right) + \mathcal{O}(\varepsilon) \quad (\text{optimal})$$

Conclusion

Remarks and perspectives

Better integrability in v : similar assumptions in $W = L^2(m(v)dv) \supset V$:

- ▶ $f_{\text{in}}(x, \cdot) \in W_v \Rightarrow f_{\text{kine}}^\varepsilon(t, x, \cdot) \in W_v$ but $f_{\text{hydro}}^\varepsilon(t, x, \cdot) \in V_v \subset W_v$
- ▶ nonlinear theory more complicated (Boltzmann without cutoff/Landau)

Constructive estimates: Enlargement theory + hypocoercivity \Rightarrow no compactness

Works for bilinear approximation of non-bilinear models:

- ▶ Boltzmann/Landau-Fermi-Dirac: $\mathcal{C}[F] = \mathcal{Q}(F, F) + \text{trilinear term}$
- ▶ BGK: $\mathcal{C}[F] = \mathcal{M}[F] - F$
- ▶ Nonlinear Fokker-Planck: $\mathcal{C}[F] = T_F \nabla_v \cdot \left(\mathcal{M}[F] \nabla_v \left(\frac{F}{\mathcal{M}[F]} \right) \right)$

Use of isotropy: adaptable to relativistic velocities $v \cdot \nabla_x \rightarrow \frac{v}{\langle v \rangle} \cdot \nabla_x$

Thank you for your attention

Extra

Kato's reduction process: eigen. pbm. in Banach \rightarrow eigen. pbm. in finite dimension

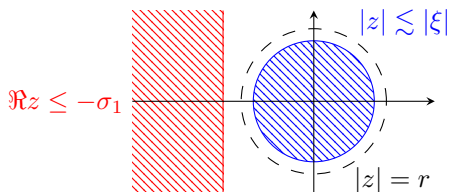


Figure: Localization of $\text{Spec}(\mathcal{L} - i(v \cdot \xi))$ for $|\xi| \ll 1$

Goal: expansion of eigenvalues and eigenfunctions of $(\mathcal{L} - i(v \cdot \xi))|_{\text{hydro. space}(\xi)}$

Difficulty: $\text{hydro. space}(\xi)$ depends on ξ

Extra

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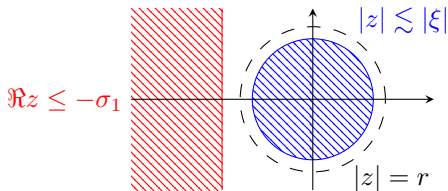


Figure: Localization of $\text{Spec}(\mathcal{L} - i(v \cdot \xi))$ for $|\xi| \ll 1$

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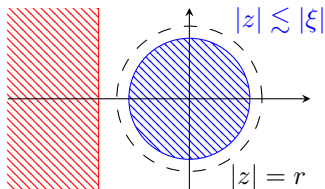
Difficulty: $\text{hydro. space}(\xi)$ depends on ξ

Solution: Rectify $\mathcal{L} - i(v \cdot \xi)$ to a matrix $L(\xi)$ by conjugating with

$$\Phi(\xi) : \text{hydro. space}(\xi) \xrightarrow{\text{iso.}} \ker \mathcal{L} \approx \mathbb{C}^{d+2}$$

Extra

Kato's reduction process



Projection on the hydrodynamic spectrum $(\xi) = R(\mathcal{P}(\xi))$:

$$\mathcal{P}(\xi) = \frac{1}{2i\pi} \oint_{|z|=r} (z - \mathcal{L} + i(v \cdot \xi))^{-1} dz \in \mathcal{B}(V^{-1} \rightarrow V^1)$$

Kato's isomorphism:

$$\frac{\mathcal{P}(0)\mathcal{P}(\xi) + \mathcal{P}(0)^\perp \mathcal{P}(\xi)^\perp}{\sqrt{\text{Id} - (\mathcal{P}(\xi) - \mathcal{P}(0))^2}} =: \Phi(\xi) : \text{hydro. space}(\xi) \xrightarrow{\text{iso}} \ker(\mathcal{L})$$

$$\Phi(\xi) = \text{Id} + |\xi|\Phi^{(1)} + |\xi|^2\Phi^{(2)}, \quad \Phi^{(j)} \in \mathcal{B}(V^{-1} \rightarrow V^1)$$

Rectified operator:

$$L(\xi) := \Phi(\xi)(\mathcal{L} - i(v \cdot \xi))\Phi(\xi)^{-1} \in \mathcal{B}(\ker \mathcal{L}) \sim \mathbb{C}^{(d+2) \times (d+2)}$$

Extra

Diagonalization of $L(\xi) = \Phi(\xi)^{-1} (\mathcal{L} - i(v \cdot \xi)) \Phi(\xi)$

Problem: $|\xi|^{-1} L(\xi) \xrightarrow{\xi \rightarrow 0}$ matrix with **non-simple** eigenvalues

Solution: Isotropy of $\mathcal{L} \Rightarrow$ block representation of $L(\xi)$:

$$L(\xi) = \begin{pmatrix} \lambda_{\text{inc}}(\xi) \text{Id} & 0 \\ 0 & |\xi| M(\xi) \end{pmatrix}, \quad \lambda_{\text{inc}}(\xi) = -\kappa_{\text{inc}} |\xi|^2 + \mathcal{O}(|\xi|^3),$$

in the decomposition

$$\{u \cdot v\mu \mid u \perp \xi\} \oplus \{\varrho\mu + \alpha\xi \cdot v\mu + e|v|^2\mu : (\varrho, \alpha, e) \in \mathbb{R}^3\}$$

where

$$M(\xi) = \begin{pmatrix} +ic & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -ic \end{pmatrix} + |\xi| \begin{pmatrix} -\kappa_{\text{osc}} & * & * \\ * & -\kappa_{\text{Fou}} & * \\ * & * & -\kappa_{\text{osc}} \end{pmatrix} + \mathcal{O}(|\xi|^2)$$

Conclusion: $L(\xi)$ diagonalizable + expansion of eigenprojector and eigenvalues:

$$\lambda_{\pm\text{osc}}(\xi) = \pm ic|\xi| - \kappa_{\text{osc}}|\xi|^2 + \mathcal{O}(|\xi|^3), \quad \lambda_{\text{Fou}}(\xi) = -\kappa_{\text{Fou}}|\xi|^2 + \mathcal{O}(|\xi|^3)$$

$$L(\xi) = \sum_{\star} \lambda_{\star}(\xi) P_{\star}(\xi), \quad P_{\star}(\xi) = P_{\star}^{(0)} + |\xi| P_{\star}^{(1)} + |\xi|^2 P_{\star}^{(2)}$$

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